National Exams December 2013

07-Mec-B6, Advanced Fluid Mechanics

3 hours duration

<u>Notes:</u>

1. If doubt exists as to the interpretation of any question the candidate is urged to submit with the answer paper a clear statement of the assumptions made.
2. Candidates may use any non-communicating calculator. The exam is OPEN BOOK.
3. Answer any <u>3</u> of the 4 questions in **PART A.**
4. Answer any <u>2</u> of the 3 questions in **PART B.**
5. Weighting: Part A: 48%, Part B: 52%. Part A: Each question is weighted (16). Part B: Each question is weighted (26).

PART A: Answer any **3** of the following 4 questions.

Question A1: A 0.4 m diameter fan running at 970 rpm is tested when the air temperature is 10°C and the barometric pressure is 772 mm Hg. The following data are observed: $Q = 0.7\ m^3/s$, fan total pressure = 25 mm H_2O, shaft power = 250 W. Find the corresponding volume flow rate, fan total pressure and shaft power of a geometrically similar fan of 1 m diameter running at 500 rpm when the air temperature is 16°C and the barometric pressure is 760 mmHg. Assume that the fan efficiency is unchanged.

Question A2: A large reservoir is filled with air at 500 kPa and 77°C. It is connected to a pipe of inner diameter 5 mm though a converging-diverging nozzle of area ratio 1.688 as shown in Fig. A2. The pipe may be assumed to be well insulated. The flow inside the convergent-divergent nozzle may be assumed isentropic (section A-B)

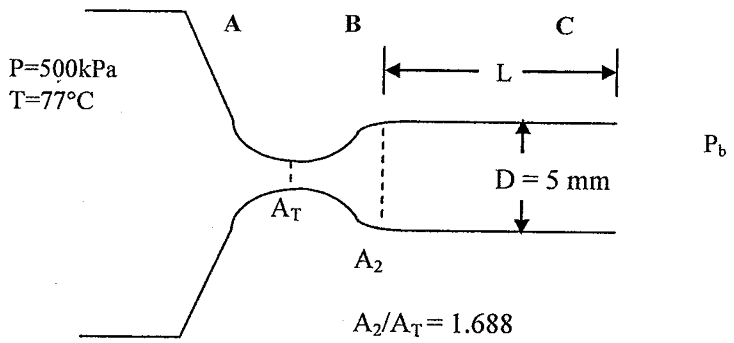


Figure A2: Basic layout for convergent-divergent nozzle attached to pipe.

If the pipe has a length of $L = 5.63$ cm and a friction factor of $f = 0.015$, determine the back pressure, P_b, if a normal shock forms at the exit of the pipe. What is the mass flow rate?

Question A3: A converging nozzle with an exit area of 1.0 cm^2 is supplied from a gas reservoir in which the stagnation pressure is 500 kPa and the stagnation temperature is 1200K. Calculate the mass flow rate for back (exit) pressures of 0, 250 and 400 kPa if the gas is Oxygen ($\gamma = 1.4$, $R = 259.8$ J/kg-K). The flow may be assumed isentropic (i.e. no frictional losses).

Question A4: A glass beaker of internal diameter 5cm is filled with glycerol (a Newtonian liquid with $\rho = 1260$ kg/m^3 and $\mu = 1.49$ Pa-s). The beaker is rotated about its centre-axis at a constant rate of 120rpm. After an initial start-up period, the fluid motion can be assumed to reach steady-state solid body rotation with negligible radial and vertical velocity components ($u_r = 0$, $u_z = 0$) and a tangential velocity component:

$$u_\theta = \omega \cdot r \ .$$

Under steady state rotation, the height of the fluid at the centre of the beaker (i.e. the axis of rotation) is 3cm.

Determine the pressure distribution along the bottom of the beaker and the equation describing the shape of the free surface. What is the maximum fluid elevation?

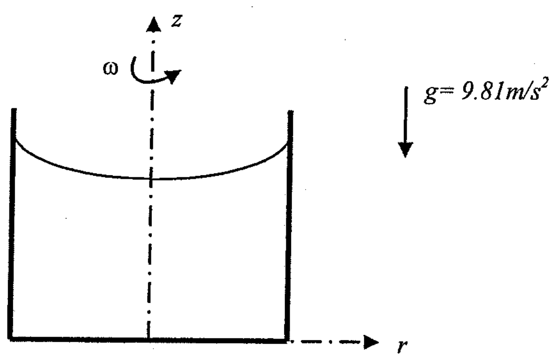


Figure A4: Schematic of a rotating beaker filled with a Newtonian Fluid.

PART B: Answer any <u>2</u> of the following 3 questions.

<u>**Question B1:**</u> A long municipal discharge pipe can be modeled as a source of strength m. The flow it induces can be modeled by the superposition of this source and a vortex of circulation, Γ, placed a distance a = 2m from the bottom of a deep lake as shown in Fig. B1. Assuming that the waste water density is the same as the lake water, that the lake bed is flat, that the fluid velocity in the lake far from the discharge is negligible and that the free-surface effects can be neglected, determine:

(a) The stream function that will represent the flow of the pipe at a distance a from the lake bed.

(b) Verify that the lake bed is effectively a streamline either by showing that the stream function has a constant value along the lake bed or, alternatively, that the vertical velocity component is identically 0 along the lake bed.

(c) What is the velocity distribution along the lake bed in terms of m and Γ?

(d) If the waste water is discharged at a constant rate of 0.5m^3/s per unit length of pipe, what is the value of Γ if the velocity at $x = 0$, $y = 0$ (i.e. underneath the pipe on the lake bed) is 0.07958 m/s?

(e) Determine the pressure distribution along the lake bed assuming that the pressure far from the discharge is P_o.

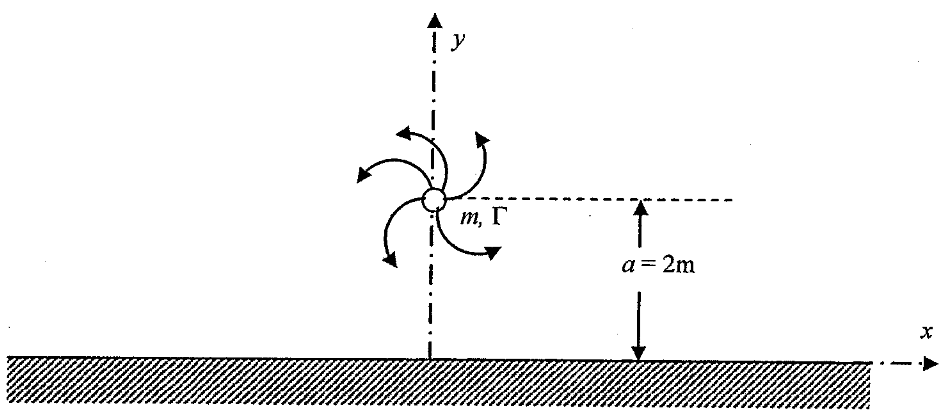


Figure B1: Cross-sectional view of a discharge pipe next to a lake bed.

Question B2: A thin film of liquid (density ρ, kinematic viscosity μ) is entrained up a flat conveyor belt as shown in Fig. B2. The belt is inclined at an angle θ, relative to the horizontal, and is moving up at a constant speed U_p. The film is exposed to air and the stress at the air-liquid interface is assumed negligible. The pressure along the film surface is atmospheric and constant. If the film thickness, δ, is constant, determine:

(a) The velocity profile in the film and the boundary conditions on the velocity field.
(b) The shear stress acting on the belt.
(c) The minimum speed U_p such that the net mass flow rate in the film is up along the belt.

It is assumed that the liquid wets the belt, that δ is small relative to the belt dimensions and that the belt is non-porous.

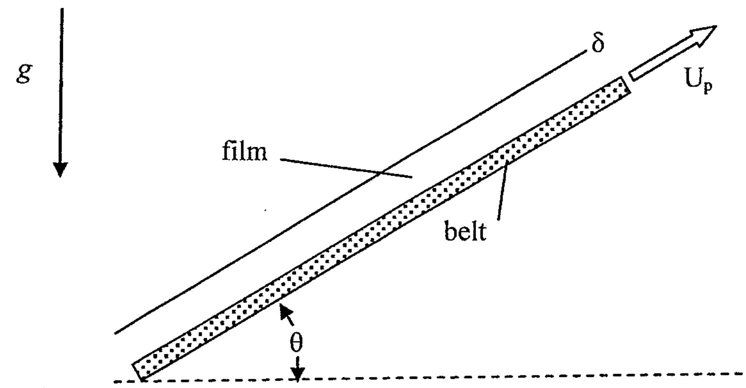


Figure B2: Inclined belt drawing a thin liquid film.

Question B3: A supersonic wind tunnel is designed with two convergent-divergent nozzles as shown in Fig. B3. The flow (air, γ =1.4, R = 287 J/kg-K) enters the test section supersonically, goes through a stationary normal shock in the test section and exits supersonically through the second nozzle. You are given:

- The cross-sectional area of the upstream throat is $A_{T1} = 10\ cm^2$.
- The test-section cross-sectional area is $A = 20\ cm^2$.
- The cross-sectional area at the exit of the second nozzle is $A = 20\ cm^2$.
- All losses, except those due to the shock, are negligible.

When the wind tunnel is operated at its design back pressure of P_b = 100 kPa, the flow exits supersonically and the static temperature, measured after the test-section shock, is 280K.

a) Determine the Mach number directly upstream and downstream of the test-section shock as well as the total (stagnation) temperature.
b) What is the cross-sectional throat area, A_{T2}, of the second nozzle? Hint: the mass flow rate is constant throughout the wind tunnel.
c) What is the air temperature and speed at the exit of the wind tunnel?
d) Determine the needed total (stagnation) pressure to drive the flow.

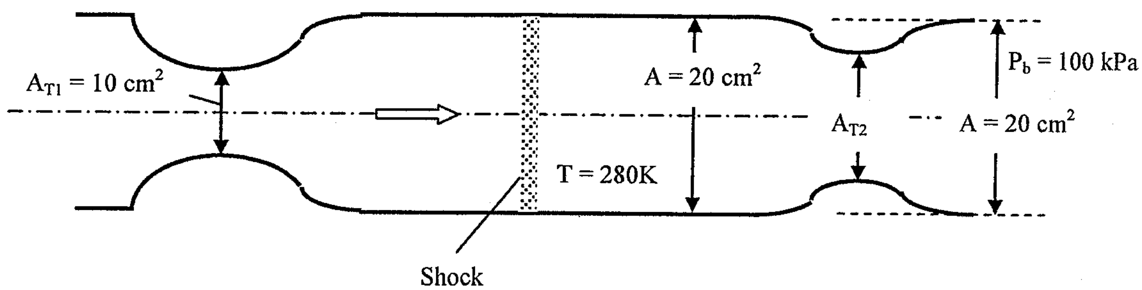


Figure B3: Schematic of wind tunnel flow with shock in test-section.

Aid Sheets

Compressible Flow:

Adiabatic flow: $$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Isentropic flow: $$\frac{P_o}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \quad ; \quad \frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

$$\dot{m} = \rho U A = \sqrt{\frac{\gamma}{R}} \cdot \frac{P_o}{\sqrt{T_o}} \cdot M \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A$$

Shock Relations: $$\frac{P_2}{P_1} = \frac{1 + \gamma M_1^{\,2}}{1 + \gamma M_2^{\,2}} = \frac{2\gamma M_1^{\,2}}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \qquad M_2^2 = \frac{M_1^2 + \dfrac{2}{\gamma - 1}}{\dfrac{2\gamma}{\gamma - 1} M_1^2 - 1}$$

Boundary Layer Equations:

$$\frac{d}{dx}\left(\mathrm{U}_\mathrm{o}^2 \theta\right) + \mathrm{U}_\mathrm{o} \frac{d\mathrm{U}_\mathrm{o}}{dx} \cdot \delta^* = \tau_w / \rho$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Laminar flow: $$C_{fx} = \frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.67}{\mathrm{Re}_x^{1/2}}$$

Turbulent flow: $$C_{fx} = \frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0266}{\mathrm{Re}_x^{1/7}}$$

Conservation Equations for Cartesian Co-ordinate system

Continuity Equation:

$$\frac{D\rho}{Dt}+\rho\left(\nabla\bullet\vec{U}\right)=0$$

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\quad\text{and}\quad\nabla\bullet\vec{U}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

Linear Momentum:

***x*-direction:** $$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}=-\frac{\partial P}{\partial x}+\rho g_x+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}$$

***y*-direction:** $$\rho\frac{\partial v}{\partial t}+\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}=-\frac{\partial P}{\partial y}+\rho g_y+\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}$$

***z*-direction:** $$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}=-\frac{\partial P}{\partial z}+\rho g_z+\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}$$

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\nabla\bullet\vec{U}\qquad\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\qquad\tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\tau_{yy}=2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\nabla\bullet\vec{U}\qquad\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\qquad\tau_{zz}=2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\nabla\bullet\vec{U}$$

$$\nabla\times\vec{U}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\cdot\vec{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\cdot\vec{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\cdot\vec{k}$$

Conservation Equations for Cylindrical-polar Co-ordinate system

Continuity Equation: $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta)+\frac{\partial}{\partial z}(\rho u_z)=0$$

Linear Momentum Equations:

***r*-momentum:**

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right]$$
$$=-\frac{\partial P}{\partial r}+\rho g_r+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta})+\frac{\partial}{\partial z}(\tau_{rz})-\frac{\tau_{\theta\theta}}{r}$$

***θ*-momentum:**

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right]$$
$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{\theta r})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{\theta\theta})+\frac{\partial}{\partial z}(\tau_{\theta z})$$

***z*-momentum:**

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right]$$
$$=-\frac{\partial P}{\partial z}+\rho g_z+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{z\theta})+\frac{\partial}{\partial z}(\tau_{zz})$$

$$\tau_{rr}=\mu\left(2\frac{\partial u_r}{\partial r}-\frac{2}{3}\nabla\cdot\vec{U}\right) \qquad \tau_{\theta\theta}=\mu\left(2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)-\frac{2}{3}\nabla\cdot\vec{U}\right)$$

$$\tau_{zz}=\mu\left(2\frac{\partial u_z}{\partial z}-\frac{2}{3}\nabla\cdot\vec{U}\right) \qquad \tau_{r\theta}=\tau_{\theta r}=\mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\tau_{rz}=\tau_{zr}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right) \qquad \tau_{\theta z}=\tau_{z\theta}=\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)$$

$$\nabla\cdot\vec{U}=\frac{1}{r}\frac{\partial}{\partial r}(r u_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)+\frac{\partial}{\partial z}(u_z)$$

$$\nabla\times\vec{U}=\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}-\frac{\partial u_\theta}{\partial z}\right)\cdot\vec{e_r}+\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\cdot\vec{e_\theta}+\left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\cdot\vec{e_z}$$

Potential Flow

Stream functions:

Uniform Flow: $\Psi = U_o y = U_o r \sin\theta$

Source Flow: $\Psi = \dfrac{m}{2\pi}\tan^{-1}\left(\dfrac{y-y_o}{x-x_o}\right) = \dfrac{m}{2\pi}\theta$

Vortex Flow: $\Psi = -\dfrac{\Gamma}{4\pi}\ln\left[(x-x_o)^2+(y-y_o)^2\right] = -\dfrac{\Gamma}{2\pi}\ln r$

Doublet Flow: $\Psi = -\dfrac{\lambda(y-y_o)}{(x-x_o)^2+(y-y_o)^2} = -\lambda\dfrac{\sin(\theta)}{r}$

Potential functions:

Uniform Flow: $\Phi = U_o x = U_o r \cos\theta$

Source Flow: $\Phi = \dfrac{m}{4\pi}\ln\left[(x-x_o)^2+(y-y_o)^2\right] = \dfrac{m}{2\pi}\ln r$

Vortex Flow: $\Phi = \dfrac{\Gamma}{2\pi}\tan^{-1}\left(\dfrac{y-y_o}{x-x_o}\right) = \dfrac{\Gamma}{2\pi}\theta$

Doublet Flow: $\Phi = \dfrac{\lambda(x-x_o)}{(x-x_o)^2+(y-y_o)^2} = \lambda\dfrac{\cos(\theta)}{r}$

Velocity relationships:

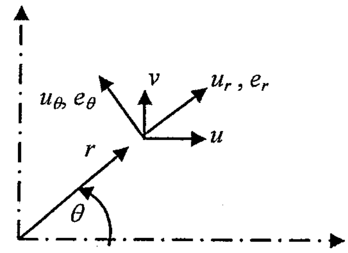
$$u = \frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad v = \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}$$

$$u_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \qquad u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r}$$

Transformation between Coordinate System

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$x = r\cos\theta$ $y = r\sin\theta$



$\vec{e}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ $\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$u_r = u\cos\theta + v\sin\theta$ $u_\theta = -u\sin\theta + v\cos\theta$

$u = u_r\cos\theta - u_\theta\sin\theta$ $v = u_r\sin\theta + u_\theta\cos\theta$