National Exams December 2002
98-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. (a) 6 marks, (b) 14 marks
2. 20 marks
3. (a) 6 marks, (b) 14 marks
4. (a) 4 marks, (b) 4 marks, (c) 12 marks
5. 20 marks
6. 20 marks
7. 20 marks
8. (a) 6 marks, (b) 14 marks

1. (a) Solve the initial value problem

$$\frac{dy}{dx} = 3x^2(1+y)^3, \qquad y(0) = 5.$$

(b) Find the general solution, $y(x)$, of the differential equation

$$y'' - 4y' + 3y = 3x^2 + e^x$$

Note that $'$ denotes differentiation with respect to x.

2. Find the general solution, $y(x)$, of the differential equation

$$y'' + 9y = \sec 3x$$

Note that $'$ denotes differentiation with respect to x.

3. (a) For $\mathbf{F} = x^2y\mathbf{i} + (z+2y)\mathbf{j} + (y-x)\mathbf{k}$, find

   i. div $\mathbf{F}$

   ii. curl $\mathbf{F}$

(b) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $2x + 3y - z = 0$, travelled counterclockwise as viewed from the positive z-axis.

4. Let $F(x,y,z) = x^2y + 3xz + z^3$, $\mathbf{u} = \sqrt{\frac{2}{3}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$, and $P = (2,-1,1)$.

(a) Find the gradient of F at the point P

(b) Find the derivative of F in the direction of $\mathbf{u}$ at the point P

(c) Find the equation of the plane tangent to the surface $F(x,y,z) = 3$ at the point P.

5. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where

$$\mathbf{F}(x,y,z) = (3x-y)\mathbf{i} + (x+3y)\mathbf{j} + 2z\mathbf{k},$$

S is the surface of the region bounded by the plane $z = 4$ and the paraboloid $z = x^2 + y^2$, and $\mathbf{n}$ is the unit outward normal on S.

6. Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$ and above the cone $z = \sqrt{x^2 + y^2}$.

7. Find the maximum and minimum values of $f(x,y,z) = 3x + 2y^2 + z$ over the ellipsoid $3x^2 + y^2 + z^2 = 1$.

8. Let $x = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 & 3 & -1 \\ -1 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

(a) Show that x is an eigenvector of A and find the associated eigenvalue.

(b) Show that 3 is an eigenvalue of A and find an associated eigenvector.