National Exams May 2003
98-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. 20 marks
2. 20 marks
3. (a) 5 marks, (b) 5 marks, (c) 10 marks
4. 20 marks
5. (a) 8 marks, (b) 12 marks
6. (a) 12 marks, (b) 4 marks, (c) 4 marks
7. 20 marks
8. (a) 3 marks, (b) 3 marks, (c) 14 marks

1. Solve the initial value problem

$$2y'' - 7y' - 4y = 3e^{4t}, \qquad y(0) = 0, \quad y'(0) = 2.$$

Note that $'$ denotes differentiation with respect to t.

2. Find the general solution, $y(x)$, of the differential equation

$$2x^2y'' + xy' - y = 3x^4.$$

Note that $'$ denotes differentiation with respect to x.

3. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ -10 & -4 & -2 \end{pmatrix}$$

(a) Show that $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of A and find the associated eigenvalue.

(b) Show that 3 is an eigenvalue of A and find and associated eigenvector.

(c) Solve the linear system $\mathbf{x}' = \mathbf{Ax}$ for the function $\mathbf{x}(t)$.

4. Find the volume of the region bounded by the paraboloid $x^2 + y^2 + z = 10$ and the plane $z = 1$ and outside the cylinder $x^2 + y^2 - 2x = 0$.

5. Let $\mathbf{A} = \begin{pmatrix} 7 & -3 \\ 15 & -5 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

(b) Solve the initial value problem

$$\frac{dx}{dt} = 7x - 3y, \qquad \frac{dy}{dt} = 15x - 5y,$$

with $y(0) = 1$ and $x(0) = 0$.

6. Consider the two lines defined as follows:
$x = 2 - t, \quad y = 3t, \quad z = 1 + t$, (parameter t);
$x = 1 + s, \quad y = 3 - 2s, \quad z = 2 + 4s$, (parameter s).

(a) Determine whether or not the two lines intersect, and if so, find the point of intersection.

(b) Find a third line orthogonal to both lines.

(c) Is there a plane containing both lines? If so, find an equation for that plane.

7. Use Lagrange multipliers to find the volume of the largest box with faces parallel to the coordinate planes that can be inscribed in the ellipsoid

$$16x^2 + 4y^2 + 9z^2 = 144.$$

8. Let S be the boundary of the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$ and let

$$\mathbf{F}(x, y, z) = xz\mathbf{i} - y^2\mathbf{j} + 2yz\mathbf{k},$$

   (a) Evaluate the divergence of $\mathbf{F}$

   (b) Evaluate the curl of $\mathbf{F}$

   (c) Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{n}$ is the unit outward normal on S.