National Exams December 2005

98-Ind-B1, Applied Probability & Statistics

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. This is a closed book exam. Candidates are permitted to use one of the two permitted calculators (Sharp or Casio models). One Aid Sheet handwritten on both sides is allowed.

3. This exam consists of five sections (A→E). Within each section, candidates will be given a choice of questions to answer. Please read the instructions for each section carefully. A breakdown of questions and marks is as follows:

Section A:	Do 2 of 3 Questions.	Total marks: 20
Section B:	Do 2 of 3 Questions.	Total marks: 20
Section C:	Do 2 of 3 Questions.	Total marks: 20
Section D:	Do 1 of 2 Questions.	Total marks: 20
Section E:	Do 1 of 3 Questions.	Total marks: 20
Exam:	8 Questions.	Total marks: 100

4. The value of each question is listed in the exam. Remember to check the instructions for each section. DO NOT ATTEMPT TO DO ALL QUESTIONS.

5. Statistical tables are provided.

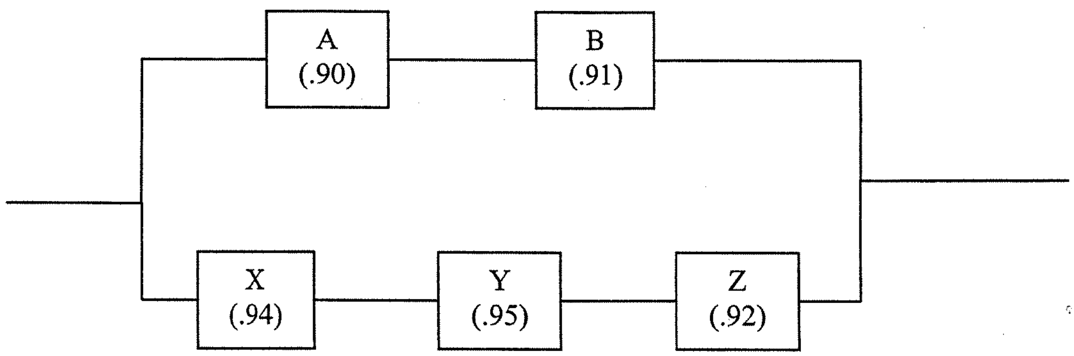
Section A: Complete two of the following three questions. This section is worth a total of 20 marks. Do not attempt all questions.

1. An urn contains 40 balls: 20 are red, 10 are blue, and 10 are white.

    i. Assume we draw three balls in a group from the urn. What is the probability that the resulting group will have 1 ball of each colour? Assume the balls in the urn are well mixed.
    ii. How many permutations of three balls are possible with the selected group of balls?

10 Marks

2. An electrical assembly has the following layout where each box represents a component with its individual uptime percentage given in brackets:



    Determine the overall uptime of the assembly.

10 Marks

3. Renfrew Real Estate (RRE) is a development company that specializes in the conversion of commercial properties to residential housing developments. RRE is considering the purchase of a property that is presently being used as an automobile service station. The existing service station will have to be removed before a housing development can be placed on the site. If the site is uncontaminated with hydrocarbons, demolition and site preparation will cost \$0.6 million. If the site is contaminated with hydrocarbons, demolition, remediation, and site preparation will total \$1.7 million.

    Historically it is known that 72% of all service stations that have been converted into residential property in Ontario have a degree of contamination that has required site remediation.

    RRE can hire Bytown Evironmental Engineering (BEE) to do a field survey of the site, prior to signing the purchase agreement. The field survey is not a perfect test however, since contamination can only be ruled out once the site is completely stripped of its underground storage tanks. BEE has produced records which indicate that in 94% of the instances its field surveys have suggested contamination, a contaminated site was eventually proven. Furthermore records show that 68% of the in which BEE indicated that a site was not contaminated were ultimately confirmed to have been uncontaminated.

If the Renfrew Real Estate orders the BEE field survey and the survey comes back negative for hydrocarbons, what is the probability that the site is actually contaminated?

10 Marks

Section B: Complete two of the following three questions. This section is worth a total of 20 marks. Do not attempt all questions.

1. A function f(x) given below:

$$f(x)=\begin{cases} x & 0<x<q \\ 2-x & q\leq x\leq 2 \\ 0 & elsewhere \end{cases}$$

   a. What value must q be for f(x) to be a valid probability function? Why?
   b. Find the mean of the function, under the assumption that it is a valid pdf.
   c. Find the variance of the function under the assumption that it is a valid pdf.
   d. Determine the standard deviation of the function under the assumption that it is a valid pdf.

10 Marks

2. Two random variables have a joint probability density function f(x,y) given by:

f(x,y)		x = 1	x = 2	x = 3
y	1	0.2	0.1	0.05
	2	0.1	0.1	0.05
	3	0.1	0.2	0.1

   a. Find the covariance of the random variables X and Y.
   b. Find the correlation of the random variables X and Y.

10 Marks

3. Let X be a random variable with the following probability distribution:

x	-3	6	9
f(x)	1/6	1/2	1/3

a. Find E(X)
b. Find Var(X)
c. Find $E[(2X + 1)^2]$
d. Find Var[(-8X + 1)]

10 Marks

Section C: Complete two of the following three questions. This section is worth a total of 20 marks. Do not attempt all questions.

1. Conestoga Confections is a Kitchener based manufacturer of potato chips. Potato chips are sold by mass. The firm's bagging machine has known characteristics when filling a 200 gram bag of potato chips. The fill is normally distributed with the following parameters: $\mu = 200.0$ grams, $\sigma = 0.316$ grams.
   i. Determine the proportion of bags that have a mass in excess of 200.2 grams.
   ii. If federal regulations require that no more that 1 bag in 1000 is underweight, what mass should the machine be set to fill to achieve this standard? Assume that the variance will not change.

10 Marks

2. Lillian Therblig is an industrial engineer in a manufacturing plant in Woodstock, Ontario. Ms. Therblig is to conduct a time study on a manual operation in the plant. Following good IE practices, Ms. Therblig undertakes a pilot study of the operation. She collects 10 samples from the floor (see below):

Sample	Time (sec)
1	61.8
2	69.9
3	73.0
4	71.8
5	65.0
6	58.0
7	53.6
8	69.8
9	45.5
10	48.0

If Ms. Therblig wishes to collect enough samples to yield a standard with a confidence interval of +/- 5 seconds, 19 times out of 20, how many more samples should she collect? State any assumptions that you make.

10 Marks

3. Port Perry Electronics manufactures fuses used in the automotive industry. The random variable X, representing the number of defective fuses in a standard box of fuses shipped to a customer, has the following probability distribution:

X	4	5	6	7
P(X=x)	0.2	0.4	0.3	0.1

a. Find the mean (μ) and the variance (σ^2) of X.
b. Assume that for quality control purposes, a sample of 36 standard boxes is selected for testing. Determine the mean and variance of the defects per box in the sample.
c. Find the probability that the average number of defects per box in a sample of 36 boxes is less than 5.0.

10 Marks

Section D: Complete one of the following two questions. This section is worth a total of 20 marks. Do not attempt all questions.

1. An athletic footwear company has invented a new smart running shoe with a microchip in the sole of the shoe. The chip senses the wearer's gait and makes small adjustments to change the shoe's stiffness and cushioning as the wearer runs. The shoe is expensive, but the firm believes that average runners' performance improves with this shoe. Accordingly, the firm recruits 20 runners who have registered for the Toronto marathon. 10 of the runners are given a pair of the smart shoes; 10 are given a pair of their competitor's product. Both groups are given a two week period to break in the shoes. The finishing times for the runners in the two groups (time in minutes) are listed below.

Smart Shoe	Competitor's Product
224.8	225.5
219.4	227.6
201.5	211.2
222.5	232.5
221.3	250.7
230.1	231.1
246.2	216.4
230.6	235.9
220.2	238.2
228.5	221.3

Can the firm support its claim that the shoe improves an average runner's performance at a significance level of 0.05? You may assume that finishing time for both populations is normally distributed. State any assumptions you make in your analysis.

20 Marks

2. A machine dispensing a liquid adhesive at a tire plant in Rockwood, Ontario has been completely overhauled recently to fix a problem in one of the heating elements. Measurements of the amount of adhesive dispensed (in ml) before and after the overhaul are listed below:

Before:	12.9	15.1	14.2	15.3	14.3	13.5	15.0	13.7
After:	17.0	19.7	11.1	18.8	10.7	19.3	12.2	17.6

a. Can you conclude at the 5% level that the variance after the overhaul is lower than the variance before the overhaul?
b. Can you conclude at the 5% level that the process mean has changed?

20 Marks

Section E: Complete one of the following three questions. This section is worth a total of 20 marks.

1. Fred Taylor is building a simulation model of a distribution centre for an Ontario based hardware/household goods retailer. Mr. Taylor has collected data on the time between order arrivals at the firm's Brampton warehouse. A total of 20 samples was collected and were sorted by inter-arrival time:

Sample	Time (Hrs)
1	0.1
2	0.1
3	0.2
4	0.3
5	0.3
6	0.4
7	0.4
8	0.5
9	0.6
10	0.6
11	0.7
12	0.7
13	0.8
14	1.0
15	1.1
16	1.2
17	1.4
18	1.5
19	1.6
20	2.2

Mean	0.78
StdDev	0.58

Mr. Taylor wishes to determine whether this sample can be adequately represented in his simulation with an exponential distribution with an arrival rate of 1 order per hour. Conduct a Chi-Squared goodness of fit test, using $\alpha = 0.05$ to determine whether this dataset is adequately represented by an exponential distribution. As you develop your analysis, please make sure that you note any assumptions you make.

As a reminder, the pdf for an exponential distribution is as follows:

$$f(x) = \lambda e^{-\lambda x}$$

where λ is the average arrival rate (in jobs per hour).

20 Marks

2. A variable y was observed at 12 different combinations of values of controlled variables x_1, x_2. The results are shown below.

x1	x2	y
-2	0.5	15
-3	-0.5	11
1	-0.5	17
4	1.5	18
3	-2.5	23
-2	-4.5	11
-3	2.5	17
-1	-3.5	13
-4	0.5	14
5	1.5	32
0	1.5	21
2	3.5	24

a. If the relation between y and (x_1, x_2) is assumed to be linear in the region of interest, find the least squares regression model for this data set. To aid you in your calculations assume the following:

$\sum x_{1i} = 0$
$\sum x_{2i} = 0$

$\sum x_{1i}^2 = 98$
$\sum y_i^2 = 4304$

$\sum x_{1i} x_{2i} = 16$
$\sum x_{1i} y_i = 161$

$\sum x_{2i} x_{1i} = 16$
$\sum x_{2i} y_i = 81$

A partial table of $(X'X)^{-1}$ is given below, where * and ** remain to be determined:

0.083333	0	0
0	0.010631	*
0	-0.00262	**

b. Complete the ANOVA table for this regression model and determine whether the linear model is a significant predictor of y. For ease of calculations, you may assume that SST = 416 and SSR = 312.48.

20 Marks

3. The University of West Meaford is conducting an analysis of its first year students. Four students who had graduated from one of six area high schools, were randomly selected and student's first year university numerical mark was recorded (see below):

High School					
1	**2**	**3**	**4**	**5**	**6**
69	82	68	70	52	90
51	77	45	94	68	46
67	82	75	74	45	67
62	79	91	78	87	87

Perform an analysis of variance at the 0.05 level of significance and indicate whether or not high school has a significant impact on a student's 1st year marks.

For ease of calculations, you may assume that SST = 5091.8

20 Marks

Critical Values of the F Distribution

	$f_{0.05(v1,v2)}$									
	v1									
v2	1	2	3	4	5	6	7	8	9	10
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91
∝	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83

Critical Values of the F Distribution

	$f_{0.05(v1,v2)}$									
	v1									
v2	10	12	15	20	24	30	40	60	120	∝
1	241.88	243.90	245.95	248.02	249.05	250.10	251.14	252.20	253.25	254.30
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.41
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
50	2.03	1.95	1.87	1.78	1.74	1.69	1.63	1.58	1.51	1.44
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.26
∝	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.01

Critical Values of the F Distribution

	$f_{0.01(v1,v2)}$									
	v1									
v2	1	2	3	4	5	6	7	8	9	10
1	4052.18	4999.34	5403.53	5624.26	5763.96	5858.95	5928.33	5980.95	6022.40	6055.93
2	98.50	99.00	99.16	99.25	99.30	99.33	99.36	99.38	99.39	99.40
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32

Critical Values of the F Distribution

$f_{0.01(v1,v2)}$

v2	v1: 10	12	15	20	24	30	40	60	120	∝
1	6055.93	6106.68	6156.97	6208.66	6234.27	6260.35	6286.43	6312.97	6339.51	6365.59
2	99.40	99.42	99.43	99.45	99.46	99.47	99.48	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.01
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.07
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.04
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.81
50	2.70	2.56	2.42	2.27	2.18	2.10	2.01	1.91	1.80	1.68
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∝	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.01