# National Examinations
# May 2003

# Applied Probability and Statistics
# Ind-B1

**Notes: a)** Solve either four or five questions whose total points equal to 100.

**b)** Distribution of points:

Q-1 15 pts
Q-2 15 pts
Q-3 20 pts
Q-4 20 pts
Q-5 30 pts
Q-6 30 pts

**c)** Candidate may use one of two calculators, Casio or Sharp.

**d)** This is a closed book exam.

**Q-1. (15 pts)** Suppose that a random experiment consists of asking television viewers in a certain city of they regularly watch three specified programs.

Let $S_1$ and $S_2$ be two events in the sample space $S$ in the random experiment stated above.

a) If "Y" stands for "yes" and "N" stands for "no" to a program, then

(i) how many points are there in the sample space, $S$?

(ii) Specify all possible outcomes in the sample space $S$ in terms of "Y" and "N".

b) In particular let $S_1$ be the event defined by the condition that "yes" is the answer for program number one.

(i) how many points are contained in $S_1$?

(ii) specify all possible outcomes in $S_1$ in terms of "Y" and "N".

c) Also let $S_2$ be the event defined by the condition that "no" is the answer for program number three.

(i) how many points are contained in $S_2$?

(ii) specify all possible outcomes in $S_2$ in terms of "Y" and "N"

d) What is the union of $S_1$ and $S_2$, i.e., $S_1 \cup S_2$=?, in terms of "Y" and "N".

e) What is the intersection of $S_1$ and $S_2$, i.e., $S_1 \cap S_2$=?, in terms of "Y" and "N".

**Q-2 (15 pts)** Let four balls be drawn from an urn containing five black, six white and seven red balls.

Let x be the number of white balls drawn and y be the number of red balls drawn.

a) What is the density of the two-dimensional random variable (x,y)?

b) What is the marginal density of x?

c) What is the marginal density of y?

**Q-3 (20 pts)** Let the joint density function $f(x,y)$ be defined over the region $0<x<2$ and $2<y<4$ as

$$\int_0^2 \int_2^4 (6-x-y)dydx = 8$$

i.e.,

$f(x,y) = 1/8(6-x-y)$, $0<x<2$, $2<y<4$.

a) If x and y are random variables having this density, what is the probability that they will fall in the region $x<1$ and $y<3$, i.e., $P(x<1, y<3)$=?

b) What is the probability that $x+y$ will be less than 3, i.e., $P(x+y<3)$=?

c) What is the probability that $y<3$, i.e., $P(y<3)$?

d) What is the probability that $x<1$ when it is known that $y<3$, $P(x<1| y>3)$=?

**Q-4 (20 pts)** Suppose that a true die is cast and a one or a two counted as a success, i.e., $p=1/3$ and $q=2/3$.

a) What is the density $f(j)$ for the total number of successes, j for a sample of 300 trials?

b) What is the probability that the number of successes will not deviate from 100 by more than 15?

c) Approximate this binomial distribution by a normal distribution and determine the probability in (b) above using the table of normal distributions.

Hint: Note that $\bar{x}$=j/300 is approximately normally distributed with mean 1/3 and variance 1/3*2/3*1/300.

**Q-5 (a) (10 pts)** Five Samples were drawn from populations assumed to be normal and assumed to have the same variance. The values of $(n-1)s^2 = \sum(x_i - \bar{x})^2$ and n, the sample size, were:

$s^2$: 40, 30, 20, 42, 50
n: 6, 4, 3, 7, 8

Final the 98 percent confidence limits for the common variance.

**Q-5 (b) (10 pts)** The random variable x is distributed normally with mean $\mu$ and variance 1. It is desired to test $H_1$: $\mu=6$ with the alternate hypothesis $H_1$: $\mu=7$.

Formulate this problem as a two-action decision problem and define the action space.

**Q-5 (c) (10 pts)** Let d be a decision function, where d is defined as follows ($\bar{x}$ is based on a random sample of size 4):

I: Take action $a_1$, that is, accept $H_1$, if $\bar{x}<7$.

II: Take action $a_2$, that is, accept $H_2$, if $\bar{x}\geq 7$.

Find P(I), P(II).

**Q-6 (30 pts)** Let a simple model be given by:

$$y_{11} = \mu + \alpha_1 + e_{11}$$
$$y_{12} = \mu + \alpha_1 + e_{12}$$
$$y_{21} = \mu + \alpha_2 + e_{21}$$
$$y_{31} = \mu + \alpha_3 + e_{22}$$

a) Write out the matrix X and the $\beta$ vector if this model is represented in the matrix form as $Y = X\beta + e$.

b) Find $X'X$

c) Show that $X'Y = \begin{pmatrix} Y_{..} \\ Y_{1.} \\ Y_{21} \\ Y_{22} \end{pmatrix}$

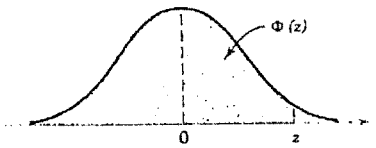
d) Write out the normal equations

e) Find the rank of $X'X$

f) Find the ML(or LS) estimate of $\alpha_1$-$\alpha_2$

Cumulative Standard Normal Distribution (*continued*)

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}}\, du$$




| z | .05 | .06 | .07 | .08 | .09 | z |
|---|---|---|---|---|---|---|
| .0 | .519 94 | .523 92 | .527 90 | .531 88 | .535 86 | .0 |
| .1 | .559 62 | .563 56 | .567 49 | .571 42 | .575 34 | .1 |
| .2 | .598 71 | .602 57 | .606 42 | .610 26 | .614 09 | .2 |
| .3 | .636 83 | .640 58 | .644 31 | .648 03 | .651 73 | .3 |
| .4 | .673 64 | .677 24 | .680 82 | .684 38 | .687 93 | .4 |
| .5 | .708 84 | .712 26 | .715 66 | .719 04 | .722 40 | .5 |
| .6 | .742 15 | .745 37 | .748 57 | .751 75 | .754 90 | .6 |
| .7 | .773 37 | .776 37 | .779 35 | .782 30 | .785 23 | .7 |
| .8 | .802 34 | .805 10 | .807 85 | .810 57 | .813 27 | .8 |
| .9 | .828 94 | .831 47 | .833 97 | .836 46 | .838 91 | .9 |
| 1.0 | .853 14 | .855 43 | .857 69 | .859 93 | .862 14 | 1.0 |
| 1.1 | .874 93 | .876 97 | .879 00 | .881 00 | .882 97 | 1.1 |
| 1.2 | .894 35 | .896 16 | .897 96 | .899 73 | .901 47 | 1.2 |
| 1.3 | .911 49 | .913 08 | .914 65 | .916 21 | .917 73 | 1.3 |
| 1.4 | .926 47 | .927 85 | .929 22 | .930 56 | .931 89 | 1.4 |
| 1.5 | .939 43 | .940 62 | .941 79 | .942 95 | .944 08 | 1.5 |
| 1.6 | .950 53 | .951 54 | .952 54 | .953 52 | .954 48 | 1.6 |
| 1.7 | .959 94 | .960 80 | .961 64 | .962 46 | .963 27 | 1.7 |
| 1.8 | .967 84 | .968 56 | .969 26 | .969 95 | .970 62 | 1.8 |
| 1.9 | .974 41 | .975 00 | .975 58 | .976 15 | .976 70 | 1.9 |
| 2.0 | .979 82 | .980 30 | .980 77 | .981 24 | .981 69 | 2.0 |
| 2.1 | .984 22 | .984 61 | .985 00 | .985 37 | .985 74 | 2.1 |
| 2.2 | .987 78 | .988 09 | .988 40 | .988 70 | .988 99 | 2.2 |
| 2.3 | .990 61 | .990 86 | .991 11 | .991 34 | .991 58 | 2.3 |
| 2.4 | .992 86 | .993 05 | .993 24 | .993 43 | .993 61 | 2.4 |
| 2.5 | .994 61 | .994 77 | .994 92 | .995 06 | .995 20 | 2.5 |
| 2.6 | .995 98 | .996 09 | .996 21 | .996 32 | .996 43 | 2.6 |
| 2.7 | .997 02 | .997 11 | .997 20 | .997 28 | .997 36 | 2.7 |
| 2.8 | .997 81 | .997 88 | .997 95 | .998 01 | .998 07 | 2.8 |
| 2.9 | .998 41 | .998 46 | .998 51 | .998 56 | .998 61 | 2.9 |
| 3.0 | .998 86 | .998 89 | .998 93 | .998 97 | .999 00 | 3.0 |
| 3.1 | .999 18 | .999 21 | .999 24 | .999 26 | .999 29 | 3.1 |
| 3.2 | .999 42 | .999 44 | .999 46 | .999 48 | .999 50 | 3.2 |
| 3.3 | .999 60 | .999 61 | .999 62 | .999 64 | .999 65 | 3.3 |
| 3.4 | .999 72 | .999 73 | .999 74 | .999 75 | .999 76 | 3.4 |
| 3.5 | .999 81 | .999 81 | .999 82 | .999 83 | .999 83 | 3.5 |
| 3.6 | .999 87 | .999 87 | .999 88 | .999 88 | .999 89 | 3.6 |
| 3.7 | .999 91 | .999 92 | .999 92 | .999 92 | .999 92 | 3.7 |
| 3.8 | .999 94 | .999 94 | .999 95 | .999 95 | .999 95 | 3.8 |
| 3.9 | .999 96 | .999 96 | .999 96 | .999 97 | .999 97 | 3.9 |

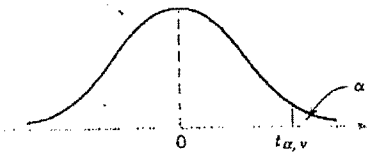



Table IV Percentage Points $t_{\alpha,\nu}$ of the t-Distribution

| $\nu$ \ $\alpha$ | .40 | .25 | .10 | .05 | .025 | .01 | .005 | .0025 | .001 | .0005 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .325 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 127.32 | 318.31 | 636.62 |
| 2 | .289 | .816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 14.089 | 23.326 | 31.598 |
| 3 | .277 | .765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 7.453 | 10.213 | 12.924 |
| 4 | .271 | .741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 5.598 | 7.173 | 8.610 |
| 5 | .267 | .727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 4.773 | 5.893 | 6.869 |
| 6 | .265 | .718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 |
| 7 | .263 | .711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 |
| 8 | .262 | .706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 |
| 9 | .261 | .703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 3.690 | 4.297 | 4.781 |
| 10 | .260 | .700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 |
| 11 | .260 | .697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 |
| 12 | .259 | .695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.428 | 3.930 | 4.318 |
| 13 | .259 | .694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 |
| 14 | .258 | .692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 |
| 15 | .258 | .691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 |
| 16 | .258 | .690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 |
| 17 | .257 | .689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 |
| 18 | .257 | .688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.197 | 3.610 | 3.922 |
| 19 | .257 | .688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 |
| 20 | .257 | .687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 |
| 21 | .257 | .686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 |
| 22 | .256 | .686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 |
| 23 | .256 | .685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.104 | 3.485 | 3.767 |
| 24 | .256 | .685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 |
| 25 | .256 | .684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 |
| 26 | .256 | .684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 |
| 27 | .256 | .684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.057 | 3.421 | 3.690 |
| 28 | .256 | .683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 |
| 29 | .256 | .683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.038 | 3.396 | 3.659 |
| 30 | .256 | .683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 |
| 40 | .255 | .681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 |
| 60 | .254 | .679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 |
| 120 | .254 | .677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 2.860 | 3.160 | 3.373 |
| ∞ | .253 | .674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807 | 3.090 | 3.291 |

$\nu$ = degrees of freedom.