National Exams May 2005
98-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. 20 marks
2. 20 marks
3. 20 marks
4. 20 marks
5. (a) 10 marks, (b) 10 marks
6. (a) 3 marks, (b) 3 marks, (c) 14 marks
7. (a) 4 marks, (b) 4 marks, (c) 12 marks
8. 20 marks

1. An elastic membrate in the x_1x_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point $P:(x_1,x_2)$ goes over into the point $Q:(y_1,y_2)$ given by

$$y_1 = 5x_1 + 3x_2,$$
$$y_2 = 3x_1 + 5x_2.$$

   Find the principal directions of the transformation. These are the directions of the position vectors $\mathbf{x}$ of all points P for which the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under the deformation?

2. Find the general solution, $y(x)$, of the differential equation

$$y'' - y' - 6y = 3x^2 + e^{-2x}$$

   Note that $'$ denotes differentiation with respect to x.

3. Find the volume of the solid region above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 6 - x^2 - y^2$.

4. Find the minimum value of the function $F(x,y,z) = 2x^2+y^2+3z^2$ subject to the constraint $x + y - z + 1 = 0$

5. Consider the two lines defined as follows:

   $x = t, \quad y = 3, \quad z = 1 - t$, (parameter t);
   $x = 2 - s, \quad y = 1 + 2s, \quad z = 2 + s$, (parameter s).

   (a) Determine whether or not the two lines intersect, and if so, find the point of intersection.

   (b) Is there a plane containing both lines? If so, find an equation for that plane, if not, find the distance between the two lines.

6. Let S be the boundary of the region enclosed by the parabola $z = 5 - x^2 - y^2$ and the plane $z = 1$ and let

$$\mathbf{F}(x, y, z) = xy\mathbf{i} - 3yz\mathbf{j} + xy^2\mathbf{k}.$$

   (a) Evaluate the divergence of $\mathbf{F}$

   (b) Evaluate the curl of $\mathbf{F}$

   (c) Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\, dA$, where $\mathbf{n}$ is the unit outward normal on S.

7. Let $F(x, y, z) = 3x^2y + y^3z + z$, $\mathbf{u} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, and $P = (2, 1, -3)$.

   (a) Find the gradient of F at the point P

   (b) Find the derivative of F in the direction of $\mathbf{u}$ at the point P

   (c) Find the equation of the plane tangent to the surface $F(x, y, z) = 6$ at the point P.

8. Find the general solution of the differential equation

$$x^2y'' - 4xy' + 6y = 3x^4.$$

   Note that $'$ denotes differentiation with respect to x.