National Examinations — May 2008

04-BS-16 — Discrete Mathematics

Duration 3 hours

NOTES

- This is a CLOSED BOOK EXAM No calculators or other aids are allowed

  All questions are to be answered in the examination booklet If you need more space use the back of the previous page and indicate clearly where your work can be found

- Ten (10) questions constitute a full paper The first ten solutions as they appear in the answer book will be marked If you want the marker to ignore a solution draw a diagonal line through it

  Each question is of equal value

  If doubt exists as to the interpretation of any question the candidate is urged to submit with the answer paper a clear statement of any assumptions made

  For full credit answers to counting problems must be expressed as integers

4 (a) (6 marks)

The *harmonic numbers* H_n are defined as follows $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \quad + \frac{1}{n}$ $n \in \mathbf{Z}^+$ where $\mathbf{Z}^+$ is the set of positive integers

Use mathematical induction to prove that $\forall n \in \mathbf{Z}^+$ $\sum_{=1}^{n} H = (n+1)H_n - n$

(b) (4 marks)

If 11 integers are chosen from the set $\{1\ 2\ 3 \quad 100\}$ use the pigeonhole principle to prove that there are at least two say x and y such that $|\sqrt{x} - \sqrt{y}| < 1$

5 (a) (5 marks)
Consider a sequence defined as follows for $n \in \mathbf{Z}^+$

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 2 \\
a_3 &= 3 \\
a_n &= a_{n-1} + a_{n-2} + a_{n-3} \text{ if } n > 3
\end{aligned}
$$

Prove that $\forall n \in \mathbf{Z}^+ \quad a_n \leq 3^{n-1}$

(b) (5 marks)
Find an explicit formula for u_n given that

$$
u_n = \begin{cases} 2 & \text{if } n = 1 \\ 7 & \text{if } n = 2 \\ u_{n-1} + 2u_{n-2} & \text{if } n > 2 \end{cases}
$$

6 (a) (3 marks)
If X ıs an m element set and Y ıs an n element set how many functıons are there from X to Y?

(b) (7 marks)
Let $\mathbf{R}^+$ denote the set of positive real numbers Let f $\mathbf{R}^+ \to \mathbf{R}^+$ and let g $\mathbf{R}^+ \to \mathbf{R}^+$ be defined as follows

$$f(x) = x^2 \qquad g(x) = \frac{1}{x}$$

i Is f a one-to one function? Justify your answer

ii Is g an onto function? Justify your answer

iii Are either of f and g bijective? Justify your answer

iv Find the compositions $f \circ f$ and g f

7 (a) (2 marks)
Arrange the following in order from fastest to slowest

$$O(n^3) \quad O(\log n) \quad O(n^2) \quad O(2^n) \quad O(1) \quad O(n \log n) \quad O(n)$$

(b) (2 marks)
Define what it means if we say that a function $f(x)$ is $O(g(x))$

(c) (6 marks)
Determine the total number of comparisons that would be required by selection sort to sort an array containing n items For full credit your answer must be written in simplified form

8 (a) (8 marks)

A relation R on a finite set S with $n \geq 1$ elements can be represented as an $n \times n$ zero one matrix A in which $a_{ij} = 1$ if and only if $i \, R \, j$

i How many relations are possible on S?

ii What property will the matrix have if R is reflexive?

iii How many relations are reflexive?

iv What property will the matrix have if R is symmetric?

v How many relations are symmetric?

(b) (2 marks)

Let R be a relation defined on the set $A = \{0 \ 1 \ 2 \ 3 \ 4\}$ such that $a \, R \, b$ if and only if $a^2 - b^2$ is divisible by 3 Find the equivalence classes of R

9 (10 marks)
How many arrangements can be made from the letters of the word REFUSAL

(a) without restriction?

(b) if the arrangement must begin and end with a vowel (A E or U)?

(c) if the arrangement must either begin or end with a consonant (F L R or S)?

(d) if all three vowels must be together?

(e) if no vowel can be next to another vowel?

10 (a) (6 marks)

Consider the experiment of choosing a committee of three people at random from among five men and four women Determine the probability of each of the following events

i Exactly two women are chosen

ii No women are chosen

iii At least one man is chosen

(b) (4 marks)

Assuming that in a four child family all sixteen possible sequences of boys and/or girls are equally likely find each of the following

i the probability that there are four girls given that the youngest child is a girl

ii the probability that there are four girls given that there is at least one girl

11 (a) (6 marks)

Consider the complete graph K_n for $n \geq 3$

i Draw a representation of K_4

ii How many edges does K_n have?

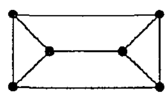
iii For what values of $n \geq 3$ does K_n have an Euler circuit?

iv For what values of $n \geq 3$ does K_n have a Hamilton circuit?

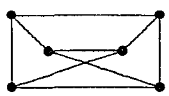
v For what values of $n \geq 3$ is K_n planar?

(b) (4 marks)

Consider the graphs G_1 and G_2 shown here



G_1

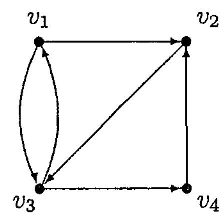


G_2

i Is G_1 bipartite? Is G_2 bipartite? (Justify your answers.)

ii Are G_1 and G_2 isomorphic? (Justify your answer.)

12 Consider the digraph shown here



(a) (2 marks)
Give an adjacency matrix A for the digraph

(b) (6 marks)
Use the matrix A to determine the number of paths of lengths 2 and 4 in the digraph

(c) (2 marks)
Is the digraph strongly connected? Justify your answer