National Exams December 2007
04-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. 20 marks
2. 20 marks
3. (a) 6 marks, (b) 14 marks
4. (a) 8 marks, (b) 12 marks
5. (a) 12 marks, (b) 4 marks, (c) 4 marks
6. 20 marks
7. (a) 4 marks, (b) 4 marks, (c) 12 marks
8. 20 marks

1. Find the general solution, $y(x)$, of the differential equation $y'' + 9y = \sec 3x$.
   Note that $'$ denotes differentiation with respect to x.

2. Find the maximum and minimum values of $f(x, y, z) = 3x + 2y^2 + z$ over the ellipsoid $3x^2 + y^2 + z^2 = 1$.

3. Let $x = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 & 3 & -1 \\ -1 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

   (a) Show that x is an eigenvector of A and find the associated eigenvalue.

   (b) Show that 3 is an eigenvalue of A and find an associated eigenvector.

4. Let $\mathbf{A} = \begin{pmatrix} 7 & -3 \\ 15 & -5 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

   (b) Solve the initial value problem $\frac{dx}{dt} = 7x - 3y$, $\frac{dy}{dt} = 15x - 5y$,
   with $y(0) = 1$ and $x(0) = 0$.

5. Consider the two lines defined as follows:
   $x = 2 - t$, $y = 3t$, $z = 1 + t$, (parameter t);
   $x = 1 + s$, $y = 3 - 2s$, $z = 2 + 4s$, (parameter s).

   (a) Determine whether or not the two lines intersect, and if so, find the point of intersection.

   (b) Find a third line orthogonal to both lines.

   (c) Is there a plane containing both lines? If so, find an equation for that plane.

6. Find the volume of the solid region above the plane $z = -4$ and below the paraboloid $z = 4 - 2x^2 - 2y^2$.

7. Let $F(x, y, z) = 3x^2y + y^3z + z$, $\mathbf{u} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, and $P = (2, 1, -3)$.

   (a) Find the gradient of F at the point P

   (b) Find the derivative of F in the direction of $\mathbf{u}$ at the point P

   (c) Find the equation of the plane tangent to the surface $F(x, y, z) = 6$ at the point P.

8. Find the general solution of the differential equation $x^2y'' - 4xy' + 6y = 3x^4$.
   Note that $'$ denotes differentiation with respect to x.