National Exams December 2010
04-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. 20 marks
2. (a) 14 marks, (b) 6 marks
3. 20 marks
4. 20 marks
5. (a) 10 marks, (b) 10 marks
6. 20 marks
7. 20 marks
8. 20 marks

The following table of antiderivatives may prove useful.

$$\int \cot x \, dx = \ln|\sin x|$$

$$\int \sec x \, dx = \ln|\sec x + \tan x|$$

$$\int \csc x \, dx = \ln|\csc x - \cot x|$$

$$\int e^{ax} \sin bx \, dx = \frac{e^a x}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^a x}{a^2 + b^2} (a \cos bx + b \sin bx)$$

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1. A tank contains 1000 litres of water in which 130 kg of salt are dissolved. Brine with a time-varying salt concentration given by the function f enters the tank at a steady rate of 50 L/min. The mixture in the tank is kept uniform by stirring and flows out at the same rate of 50 L/min. Find the amount of salt, $y(t)$ in the tank at time t given that f is defined by $f(t) = 0.02\,(1 + \cos\ t/5)$ kg/L. You must set up and solve the appropriate differential equation, but you do not need to simplify your answer.

2. Solve the following differential equations

   (a) $y'' + 4y = \csc 2x$, with $y'(0) = 0$, $y(0) = 1$.

   (b) $y' - 2y - y^2 = 0$.
   Note that $'$ denotes differentiation with respect to x.

3. Find the equation of motion of the mass-spring system corresponding to the following equation and initial conditions:

$$y'' + 2y' + 2y = \cos(t), \qquad y(0) = 1.2,\ y'(0) = 1.4.$$

   Note that $'$ denotes differentiation with respect to x.

4. Find the minimum value of the function $F(x, y, z) = 2x^2 + y^2 + 3z^2$ subject to the constraint $x + y - z + 1 = 0$

5. (a) Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

(b) Solve the system of differential equations

$$\frac{dx}{dt} = 4x + 2y,$$
$$\frac{dy}{dt} = 3x - y + e^{-2t}.$$

6. Find the work done by the field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ in moving a particle from the point $(0, 2, 0)$ to the point $(3\pi, 0, 2)$ along the path $x = 6t$, $y = 2\cos t$, $z = 2\sin t$.

7. Find the equation of the plane tangent to the surface defined implicitly by $xy^2z^3 = 2 - y$ at the point $(x, y, z) = (-3, 4, 1/2)$

8. Let S be the surface of the region defined by $x^2 + 4y^2 \leq 1$, $x \geq 0$, $y \geq 0$, $0 \leq z \leq 4$, and let F be the vector function $F(x, y, z) = (y^3, x^3, z^3)$. Evaluate the integral of F over the surface S.