National Exams May 2011
04-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. 20 marks
2. 20 marks
3. 20 marks
4. 20 marks
5. 20 marks
6. 20 marks
7. 20 marks
8. 20 marks

1. Compute the response of the damped mass-spring system modelled by

$$y'' + 3y' + 2y = r(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where r is the square wave

$$r(t) = \begin{cases} 1, & 1 \leq t < 2, \\ 0, & \text{otherwise,} \end{cases}$$

and $'$ denotes differentiation with respect to time.

2. Solve the initial value problem

$$t^2 y'' - 4ty' + 6y = \pi^2 t^4 \sin \pi t, \qquad y(1) = 5, \qquad y'(1) = 5 + \pi,$$

where $'$ denotes differentiation with respect to t.

3. Find the general solution, $y(x)$, of the differential equation

$$y'' + 2y' + 2y = 3e^{-x} \cos 2x,$$

where $'$ denotes differentiation with respect to x.

4. An elastic membrate in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P : (x_1, x_2)$ goes over into the point $Q : (y_1, y_2)$ given by

$$y_1 = 5x_1 + 3x_2,$$
$$y_2 = 3x_1 + 5x_2.$$

Find the principal directions of the transformation. These are the directions of the position vectors $\mathbf{x}$ of all points P for which the direction of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under the deformation?

5. Evaluate the surface integral $\displaystyle\iint_S \mathbf{F} \cdot dS$, where

$$\mathbf{F}(x, y, z) = 4x\mathbf{i} + 2x^2\mathbf{j} - 3\mathbf{k},$$

S is the surface of the region bounded by the cone $z = 4 - \sqrt{x^2 - y^2}$ and the plane $z = 0$.

6. Let C be the curve formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 1 + y$, and let $\mathbf{v}$ be the vector function $\mathbf{v} = 4z\mathbf{i} - 2x\mathbf{j} + 2x\mathbf{k}$. Evaluate the line integral $\oint_C \mathbf{v} \cdot d\mathbf{r}$. Assume a clockwise orientation for the curve when viewed from above.

7. Find a formula for the plane tangent to the surface $z = f(x, y)$ with $f(x, y) = 1 + x\ln(xy - 5)$ at the point $(2, 3)$ and use the tangent plane to approximate $f(1.9, 3.05)$.

8. Find the minimum value of the function $F(x, y, z) = 2x^2 + y^2 + 3z^2$ subject to the constraint $x + y - z = 7$.